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Shallow water flow down an inclined open channel: analytical solutions of governing equations .

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Abstract: In this paper we consider a problem of a fluid flow with free surface in variable turbulent mode in a long inclined open channel, taking into account the Chezy drag. We have obtained a family of exact solutions for the governing equations, i.e. a family of velocity and height fields of water above the channel bed. Moreover, we have estimated the extremum of the water height. These results can be used as essential tools for the planning and management for sustainable use of water resources(the drainage of surface waters or irrigation water of the soil in agriculture and so on).

keywords: Shallow-water equations; open channels; ordinary differential equation, analytical solution.

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1 Introduction

The rotating shallow-water model (RSW) ([5], [6], [9]) arises from the three-dimensional rotating incompressible Euler equations ([3], [9]) under the following assumptions:

(i) the scale H_o of the vertical motion is much less than the scale L_o of the horizontal motion, so that $H_o/L_o << 1$;

(ii) the fluid density ρ is constant;

(iii) the external force is gravity and the pressure obeys the hydrostatic approximation

$$p = \rho g(h - z) + p_o \, , \qquad (1)$$

p_o being the constant pressure on the free surface ;

(iv) the axis of the rotation of the fluid coincides with the vertical $z-$ axis.

The nonlinear system of partial differential equations (PDEs) is a widely used approximation for atmospheric and oceanic motions in the mid-latitudes with relatively large length scales and timescales. The model is applied to phenomena that do not substantially depend on temporal changes of the density stratification. In spite of its simplicity, it contains all essential ingredients of atmosphere and ocean dynamics at the synoptic scale. The basic mathematical results and physical applications concerning the RSW equations are presented in [1] and [3].0ne of the known methods for studying PDEs is the group analysis approach. This analytical approach based on symmetries of differential equations was originally introduced by Sophus Lie and further developed by many other authors. See [2],[4], [7] and [11] and references therein. For each system of PDEs there is a symmetry group that acts on the space of its independent and dependent variables, leaving the form of the system unchanged. Exact solutions of nonlinear systems descriptive of fluid motions with moving boundaries are uncommon even in the shallow- water (SW) approximation. In a series of remarkable papers [6], Ball found exact solutions for the finite motion of a shallow rotating liquid lying on a paraboloid. Two types of solutions were obtained: (a) pure displacement, where the liquid is displaced laterally without distortion of the depth profile and the horizontal velocity is a function of time alone; (b) simple distortion and rotation, where the centre of gravity of the liquid remains stationary and the horizontal velocity is a linear function of the space coordinates. In a subsequent paper [14], Thacker derived some exact solutions corresponding to time-dependent motion in a parabolic basin. For the case in which parabolic basin is reduced to a flat plane, solution for a flood wave was obtained. However, most of the solutions [14] are implicit in the work by Ball [5]. Sachdev et al. [13] presented a rather general analytical and numerical study of the RSW equations. Following an approach similar to that of Clarkson and Kruskal [10], in [13] exact solutions of the model such that the velocity components are linear in the spatial coordinates while the free surface is a quadratic function of the same were derived. Time - periodic solutions of the governing equations were found numerically. In a recent paper [9], Chesnok[illegible]s studied symmetry properties and some classes of exact solutions of the RSW model using group analysis; he has discovered that the RSW equations can be transformed into the usual SW model. By virtue of this result, it is possible to construct and study solutions of the RSW equations using solutions of the SW model and vice versa.

In this paper, we investigate classes of exact solutions for the SW equations [8]. In section 2, we present the shallow-water model. In section 3, we use the change of variables transforming the SW model into the ordinary differential equations (ODEs) that we integrate. Then we obtain a family of exact solutions. Furthermore we estimate the extremum of the height of water in inclined open channels. Then follows conclusion in section 4.

2 Shallow water model

The starting point of our study is the shallow-water(SW) equations governing one-dimensional flow down inclined channels with transversal rectangular section, which are non-dimensionalized with respect to the steady uniform solutions as follows:

$$H = \frac{h}{h_o}, V = \frac{v}{c_o}, \xi = \frac{x}{L}, \tau = \frac{c_0 t}{L} \qquad (2)$$

where $c_0 = c \;\; h_0$ is the wave speed of small-amplitude pressure waves given by

$$c(h) = \sqrt{gh \sin \phi} \qquad (3)$$

The non-dimensional equations are therefore as follows [8]:

$$\frac{\partial H}{\partial \tau} + \frac{\partial (HV)}{\partial \xi} = 0 \qquad (4)$$

$$\frac{\partial V}{\partial \tau} + V\frac{\partial V}{\partial \xi} + \frac{\partial H}{\partial \xi} - \frac{gL}{c_0^2}\sin\phi + R_w(H,V) = 0 \qquad (5)$$

Where H is the water height above the channel bed, V is the fluid velocity , ϕ is the angle of inclination of the channel to the horizontal, and L is the channel length. $R_w(H,V)$ is the Chezy drag, which is a measure of viscous resistance to the turbulent flow and is given by

$$R_w(H,V) = \frac{C_f L}{h_0}\frac{V^2}{H} \qquad (6)$$

Where C_f is a non-dimensional drag coefficient.
The functions V, H satisfy the initial and boundary conditions

$$H(\xi,\tau)\big|_{\tau=0} = H(\xi,\tau)\big|_{\xi=0} = H_c\,,$$
$$V(\xi,\tau)\big|_{\tau=0} = V(\xi,\tau)\big|_{\xi=0} = V_c\,;\ H_c, V_c = const \qquad (7)$$

3 Analytical solutions

In this section we provide analytical solutions to shallow-water equations. We assume the solutions belong to the space of functions that are continuously differentiable with respect to (ξ,τ) on $[0,1]\times[0,1]$, i.e. shortly denoted here by

$$V(\xi,\tau) \in \mathbb{C}^{1,1}_{\xi,\tau}([0,1]\times[0,1]),$$
$$H(\xi,\tau) \in \mathbb{C}^{1,1}_{\xi,\tau}([0,1]\times[0,1]). \qquad (8)$$

Consider the following change $z = a\tau + b\xi$ and $V(z) = V(\xi,\tau)$, $H(z) = H(\xi,\tau)$, where a, b are real parameters.
Brook et al. [8] merely summarized the findings of Dressler (1949): under the change of variable $z = \xi - c_w\tau$, i .e. $a = -c_w$, $b = 1$ Dressler transformed (4), (5) into the form

$$(H^3 - K_c^2)\frac{dH}{dz} = \frac{gL\sin\phi}{c_0^2}(H^3 - \rho c_w^2 H^2 - 2\rho c_w K_c H - \rho K_c^2)\ ,$$
$$K_c = H_c(c_w - V_c) \qquad (9)$$

$$V_c = \frac{c_w}{1+\sqrt{\rho}}\ ,\ H_c = \frac{\rho c_w^2}{(1+\sqrt{\rho})^2} \qquad \rho = \frac{C_f c_0^2}{g h_0 \sin\phi} \qquad (10)$$

and assuming that $H/_{z=0} = H_c$ is a common root of $H^3 - K_c^2$ and $H^3 - \rho c_w^2 H^2 - 2\rho c_w K_c H - \rho K_c^2$
where c_w is the speed of propagation of the roll waves, he obtained the solutions of (4), (5) in the form

$$\frac{gL\sin\phi}{c_0^2}z = H(z) - H_c + \frac{H_1^2 + H_c H_1 + H_c^2}{H_1 - H_2}\,ln\left(\frac{H(z) - H_1}{H_c - H_1}\right)$$
$$-\frac{H_2^2 + H_c H_2 + H_c^2}{H_1 - H_2}\,ln\left(\frac{H(z) - H_2}{H_c - H_2}\right), \qquad (11)$$

H_1, H_2 are roots of

$$H^2 + (H_c - \rho c_w^2)H + \rho H_c^2 = 0 \qquad (12)$$

In this paper we can state the following result which generalizes the result of Dressler

Theorem 3.1. Assume that $a < 0$ and $b > 0$. the (a,b)-family of solutions to (4), (5) satisfying (7) co (7) are defined by the functions

$$H_{(a,b)}(z)\big(bV_{(a,b)}(z) + a\big) = K_c\ ,\ K_c = H_c|a + bV_c| \qquad (13)$$

$$\frac{Lg\sin\phi}{bc_0^2}z = H_{(a,b)}(z) - H_c + A\ln\left|\frac{H_{(a,b)}(z) - H_e}{H_c - H_e}\right|$$
$$+\frac{B}{2}\ln\left|\frac{H_{(a,b)}^2(z) + (H_e - \rho a^2)H_{(a,b)}(z) + H_e^2 - \rho a^2 H_e + 2\rho a K_c}{H_c^2 + (H_e - \rho a^2)H_c + H_e^2 - \rho a^2 H_e + 2\rho a K_c}\right|$$
$$+\frac{1}{2\delta}[B(-H_e + \rho a^2) + 2C]\left\{arctan\left(\frac{2H_{(a,b)}(z) + H_e - a^2\rho}{2\delta}\right) - arctan\left(\frac{2H_c + H_e - a^2\rho}{2\delta}\right)\right\} \qquad (14)$$

Where

$$H_e = \frac{\rho a^2}{3} + \sqrt[3]{\frac{-q}{2} + \sqrt{Q}} + \sqrt[3]{\frac{-q}{2} - \sqrt{Q}}\ ,$$
$$p = -\frac{\rho^2 a^4}{3} + 2\rho a K_c, \qquad (15)$$

$$q = \frac{-2\rho^3 a^6}{27} + \frac{2}{3}\rho^2 a^3 K_c - \rho K_c^2,$$
$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -\frac{1}{27}\rho^3 a^3 K_c^3 + \frac{1}{4}\rho^2 K_c^4, \qquad (16)$$

$$\rho = \frac{C_f c_0^2}{b^2 g h_0 \sin\phi},$$

$$A = \frac{\rho a^2 H_e^2 - 2\rho a K_c H_e + \left(\rho - \frac{1}{b^2}\right)K_c^2}{3H_e^2 - 2\rho a^2 H_e + 2\rho a K_c}, \qquad B = a^2\rho - A,$$

$$C = -2a\rho K_c + a^2\rho H_e + (a^2\rho - 2H_e)A,$$

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$$\delta^2 = \frac{1}{4}(3H_e^2 - 2\rho a^2 H_e - \rho^2 a^4 + 8\rho a K_c). \quad (17)$$

Proof 3.1. *Indeed, using the change of variable* $z = a\tau + b\xi$ *and taking into account (7) we transform (4), (5) into the following relations*

$$H_{(a,b)}(z)\left(bV_{(a,b)}(z) + a\right) = K_c \quad (18)$$

$$\left(H^3 - \frac{K_c^2}{b^2}\right)\frac{dH}{dz} = \frac{gL\sin\phi}{bc_0^2}(H^3 - \rho a^2 H^2 + 2\rho a K_c H - \rho K_c^2) \quad (19)$$

Where
$K_c = H_c(a + bV_c) \geq 0$ for $\frac{-a}{b} \leq V_c$ or
$-K_c = H_c(a + bV_c) \leq 0$ for $\frac{-a}{b} \geq V_c$
equivalent to $K_c = H_c\left|a + bV_c\right|$ *. (19) can be written in the form*

$$\left\{1 + \frac{A}{H(z) - H_e} + \frac{BH(z) + C}{H^2(z) + (H_e - \rho a^2)H(z) + H_e^2 - \rho a^2 H_e + 2\rho a K_c}\right\}\frac{dH}{dz} = \frac{Lg\sin\phi}{bc_0^2} \quad (20)$$

where A, B, C, H_e *are mentioned above. The real root* H_e *of*

$$H^3 - \rho a^2 H^2 + 2\rho a K_c H - \rho K_c^2 = 0 \quad (21)$$

exists if and only if $Q > 0$*; this condition is satisfied for* $a < 0$. *Integrating (20) we obtain the solutions* $H_{(a,b)}$ *and* $V_{(a,b)}$ *satisfying (7):*

$$\frac{Lg\sin\phi}{bc_0^2}z = H_{(a,b)}(z) - H_c + A\ln\left|\frac{H_{(a,b)}(z) - H_e}{H_c - H_e}\right|$$

$$+\frac{B}{2}\ln\left|\frac{H_{(a,b)}^2(z) + (H_e - \rho a^2)H_{(a,b)}(z) + H_e^2 - \rho a^2 H_e + 2\rho a K_c}{H_c^2 + (H_e - \rho a^2)H_c + H_e^2 - \rho a^2 H_e + 2\rho a K_c}\right|$$

$$+\frac{1}{2\delta}[B(-H_e + \rho a^2) + 2C]\left\{arctan\left(\frac{2H_{(a,b)}(z) + H_e - a^2\rho}{2\delta}\right) - arctan\left(\frac{2H_c + H_e - a^2\rho}{2\delta}\right)\right\}. \quad (22)$$

that completes the proof.

Remark 3.1.
Using $H_c(a + bV_c) = K_c \geq 0$ *for* $-\frac{a}{b} \leq V_c$ *and taking into account the continuity of the functions* V *and* H *on* $[a, b]$ *we obtain* $H(z)\left(a + bV(z)\right) = K_c$ *for* $-\frac{a}{b} \leq V(z)$.
Using $H_c(a + bV_c) = -K_c \leq 0$ *for* $-\frac{a}{b} \leq V_c$ *and taking into account the continuity of the functions* V *and* H
On $[a, b]$ *we obtain* $H(z)\left(a + bV(z)\right) = -K_c$ *for* $-\frac{a}{b} \geq V(z)$.
In order to estimate the extremum of the function H we obtain the following result

Theorem 3.2. *For fixed* τ, a, b *the function* $H_{(a,b)}: \xi \mapsto H_{(a,b)}(\xi)$ *reaches on* $]0,1[$

- *a minimum* $H_{(a,b)}(\xi_1) = H_e$ *and a maximum* $H_{(a,b)}(\xi_0) = \sqrt[3]{\frac{K_c^2}{b^2}}$ *when* $\xi_0 < \xi_1$,
- *a minimum* $H_{(a,b)}(\xi_0) = \sqrt[3]{\frac{K_c^2}{b^2}}$ *and a maximum* $H_{(a,b)}(\xi_1) = H_e$ *when* $\xi_1 < \xi_0$.

Proof 3.2. *Indeed, using the equation*

$$\left(H^3(\xi) - \frac{K_c^2}{b^2}\right)\frac{dH}{dz}(\xi) = \frac{gL\sin\phi}{bc_0^2}(H^3(\xi) - \rho a^2 H^2(\xi) + 2\rho a K_c H - \rho K_c^2) \quad (23)$$

We obtain the derivative of H

$$\frac{dH}{d\xi}(\xi) = \frac{gL\sin\phi}{c_0^2}\frac{H(\xi) - H_e}{H(\xi) - \sqrt[3]{\frac{K_c^2}{b^2}}}\frac{H^2(\xi) + (H_e - \rho a^2)H(\xi) + H_e^2 - \rho a^2 H_e + 2\rho a K_c}{H^2(\xi) + \sqrt[3]{\frac{K_c^2}{b^2}}H(\xi) + \left(\sqrt[3]{\frac{K_c^2}{b^2}}\right)^2}.$$

As
$H^2(\xi) + (H_e - \rho a^2)H(\xi) + H_e^2 - \rho a^2 H_e + 2\rho a K_c > 0$ and
$H^2(\xi) + \sqrt[3]{\frac{K_c^2}{b^2}}H(\xi) + \left(\sqrt[3]{\frac{K_c^2}{b^2}}\right)^2 > 0,$
the sign of $\frac{dH}{d\xi}(\xi)$ *is similar to that of* $\frac{H(\xi) - H_e}{H(\xi) - \sqrt[3]{\frac{K_c^2}{b^2}}}$. *Then we obtain*

$$\begin{aligned}
&\frac{dH}{d\xi}(\xi) > 0 \; if \; H(\xi) > H_e \text{ and } H(\xi) > \sqrt[3]{\frac{K_c^2}{b^2}} \\
&\frac{dH}{d\xi}(\xi) > 0 \; if \; H(\xi) < H_e \text{ and } H(\xi) < \sqrt[3]{\frac{K_c^2}{b^2}} \\
&\frac{dH}{d\xi}(\xi) < 0 \; if \; H(\xi) > H_e \text{ and } H(\xi) < \sqrt[3]{\frac{K_c^2}{b^2}} \\
&\frac{dH}{d\xi}(\xi) < 0 \; if \; H(\xi) < H_e \text{ and } H(\xi) > \sqrt[3]{\frac{K_c^2}{b^2}}
\end{aligned} \quad (24)$$

Equivalently

$$\frac{dH}{d\xi}(\xi) > 0 \; if \; \xi > \xi_1 \text{ and } \xi > \xi_0$$

$$\frac{dH}{d\xi}(\xi) > 0 \; if \;\; \xi < \xi_1 \; and \; \xi < \xi_0$$

$$\frac{dH}{d\xi}(\xi) < 0 \; if \;\; \xi < \xi_1 \; and \; \xi > \xi_0$$

$$\frac{dH}{d\xi}(\xi) < 0 \; if \;\; \xi > \xi_1 \; and \; \xi < \xi_0$$

where $H(\xi_1) = H_e, \; H(\xi_0) = \sqrt[3]{\frac{K_c^2}{b^2}}$.

Finally we obtain that H *reaches its minimum* H_e, *its maximum* $\sqrt[3]{\frac{K_c^2}{b^2}}$ *for* $\xi_0 < \xi < \xi_1$ *and its minimun* $\sqrt[3]{\frac{K_c^2}{b^2}}$, *its maximum* H_e, *for* $\xi_1 < \xi < \xi_0$; *that ends the proof.*

4 Conclusion

In this paper, we have succeeded in transforming the shallow water equations into ordinary differential equations which have been integrated to obtain a family of exact solutions, generalizing the Dressler solution. Furthermore we estimated the extremum of the water height. The knowledge of the velocity and the extremal values of the water height is useful and important for the estimation of the average capacity of the channel and the prediction of the dimensions of infrastructures required for the drainage of surface waters, irrigation water of the soil in agriculture.

References

[1] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean. *Courant Institute of Mathematical Sciences*. New York. 2003.

[2] P. J. Olver, Applications of Lie Groups to Differential Equations. *Springer.* New York. 1993.

[3] Pedlosky, J. Geophysical Fluid Dynamics. *Springer.* New York. 1979.

[4] C. Rogers and W. F. Ames, Nonlinear Boundary Value Problems in Science and Engineering. *Academic Press.* New York. 1989.

[5] F. K. Ball, Some general theorems concerning the finite motion of a shallow rotating liquid lying on a paraboloid. *J.Fluid Mech.* 17, 240-256. 1963.

[6] F. K. Ball, The effect of rotation on the simpler modes of motion of a liquid in an elliptic paraboloid. *J.Fluid Mech.* 22, 529-545. 1965.

[7] N. Bila, E. Mansfield and P. Clarkson, Symmetry group analysis of the shallow water and semi-geostrophic equations. *Quart. J. Mech. Appl. Math.* 59, 95-123. 2006.

[8] B. S. Brook, S. A. Falle and T. J. Pedley, Numerical solutions for unsteady gravity-driven flows in collapsible tubes : evolution and roll-wave instability of a steady state. *J. Fluid Mech.* 396, 223-256. 1999.

[9] A.A. Chesnokov, Symmetries and exact solutions of the rotating shallow- water equations. *Euro.Jnl of Applied Mathematics*, 20,461- 477. 2009.

[10] P. A. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation. J. *Math. Phys.* 30, 2201-2213. 1989.

[11] L.V. Ovsyannikov, Optimal systems of subalgebı *Dokl.Akad. Nauk.* 333, 702-704. 1993.

[12] A. S. Pavlenko, Symmetries and solutions of equations of two-dimensional motions of polytropic gas. *Siberian Electro Math.* Rep. 2, 291-307. 2005.

[13] P. L. Sachdev, D. Palaniappan and R. Sarathy, Regular aı... chaotic flows in paraboloid basin and eddies. *Chaos Solit.Fract.* 383-408. 1996.

[14] W.C. Thacker, Some exact solutions to the nonlinear shallow water wave equations. *J. Fluid Mech.* 107,499-508. 1981.

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